Numerical Simulations of Kepler's Laws in Excel

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Euler's Method

The use of numerical methods to solve problems in orbital mechanics is a topic that is not often covered in introductory physics, although it is seen in the Feynman Lecture Series[1]. However, the widespread use of Excel in physics labs is an underutilized tool that can be used to introduce students to numerical approaches to the solution of complicated problems.

It is instructive to start with a one-dimensional problem. Four columns are created: time, **x**(t), **v**(t) and **a**(t). Each row represents a step in time, so the time column increments by a user-defined step size, **Δt**. To start, assume a constant acceleration, although this need not be the case. As the acceleration is given as **Δv**/**Δt**, this recovers the classic constant-acceleration kinematic formula of $\mathbf{v_f}=\mathbf{v_i}+\mathbf{a}\Delta t$. For an non-constant acceleration, the assumption made is that over the small **Δt** the acceleration can be treated as if it was constant.

With knowledge of row n, the velocity in row n+1 can be calculated: $\mathbf{v_{n+1}}=\mathbf{v_n}+\mathbf{a_n}\Delta t$. This assignment is trivial in a spreadsheet. Position is determined by $\Delta\mathbf{x}=\mathbf{v}\Delta t$, or $\mathbf{x_{n+1}}=\mathbf{x_n}+\mathbf{v_n}\Delta t$. A slightly better result can be found by evaluating **v** at the current iteration rather than the previous, $\mathbf{x_{n+1}}=\mathbf{x_n}+\mathbf{v_{n+1}}\Delta t$. This twist is the Euler-Cromer method, details can be found in numerical analysis texts[2].

The Euler-Cromer method can be easily extended into two dimensions simply by adding columns for **y**(t), $\mathbf{v_y}$(t) and $\mathbf{a_y}$(t). The method gains its true power when **a** is varied. This allows the introduction of twists such as forces that are functions of position or velocity, which are traditionally very difficult to examine. A classic example is gravity. Assuming a massive object at the origin, $F(x,y) = GMm/r^2$, pointed towards the origin. Broken down into x and y components:

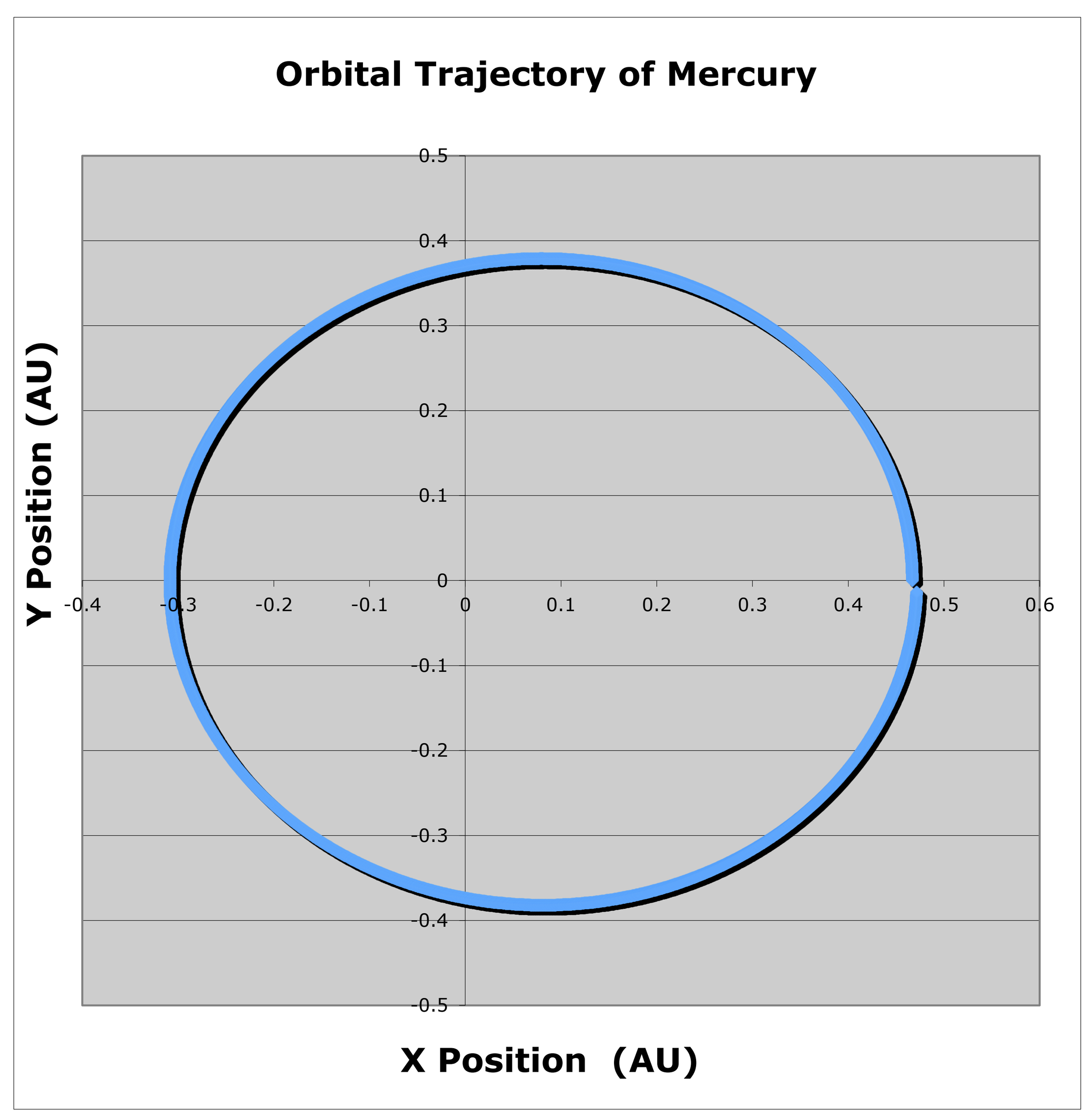
$F_x= GMm(-x/r^3)$ and $F_y = GMm(-y/r^3)$

Dividing through by m gives the accelerations as a function of position. Given starting positions and velocities, the spreadsheet shows the time evolution of the system. This tool can be used to fully explore Kepler's Laws.

The First Law

I - *The orbit of every planet is an ellipse with the Sun at one of the two foci.*

Given a good choice of initial conditions, students can quite easily generate ellipses. Note that they will need to pick a combination of speeds and distances that results in a negative total energy, otherwise hyperbolic orbit will be generated. While there is nothing wrong with a comet on an escape trajectory, it isn't described by the First Law. Additionally, orbits that tend too near to the central object will be subject to very large forces near the origin relative to other parts of the orbit, and this will cause some of the approximations to generate large errors. A possible exercise is to have students pick actual planetary data as a starting point. Mercury is shown below.

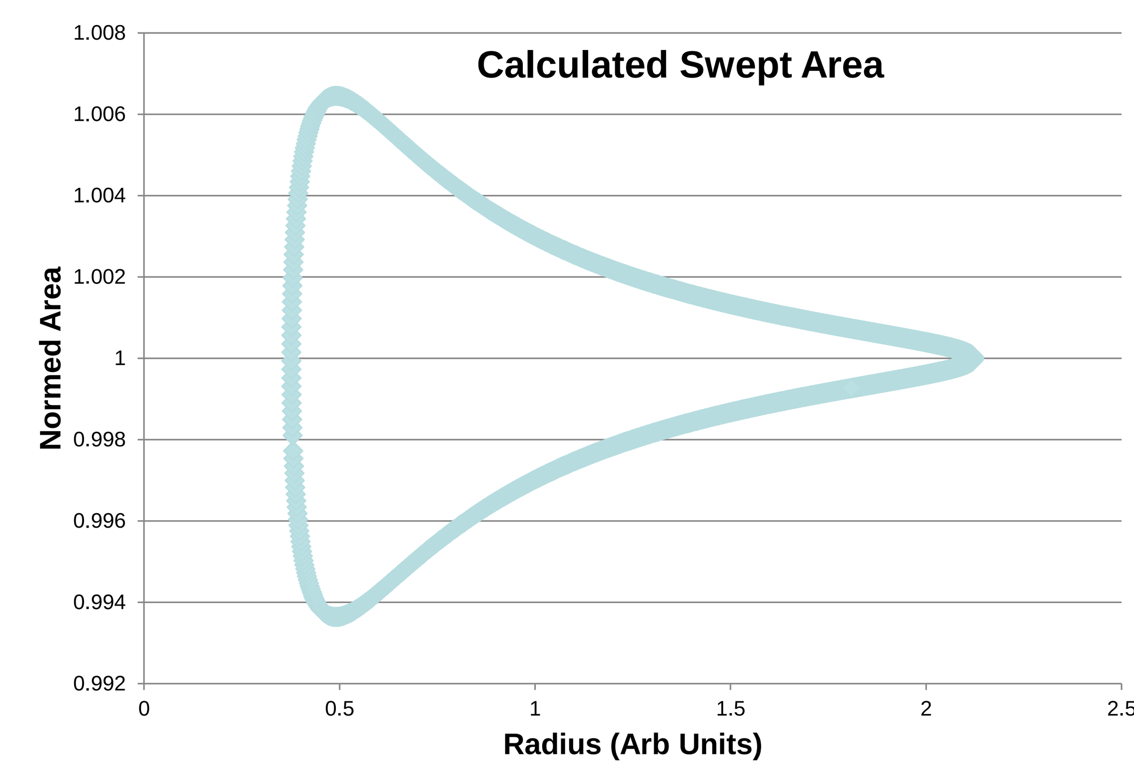


Given time constraints, students are asked to generate orbits with different eccentricities and to visually confirm that the orbits are ellipses. However, a more careful treatment would have students confirm that the orbits actually are elliptical with the origin at one of the foci. This can lead to good analytic geometry discussions.

The Second Law

II - *A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.*

This result can be calculated directly by approximating the area swept as a triangle with a height of **r** and a base of r**Δθ**. The graph below shows the results for an ellipse with 3000 time steps, and an r ranging from 0.4 to 2.1, with the swept area changing by roughly 1% between the extremes:



The small discrepancy in the swept area occurs as the orbiting object gets closer to the central mass, and as the object gets faster the approximation gets worse. An alternative approach is to calculate the angular momentum as a function of position, as the triangular area approximation approaches a multiple of the angular momentum in the limit Δt➔0. This can be done by directly evaluating the cross product of the position and velocity: $\mathbf{xv_y} - \mathbf{yv_x}$. This quantity is constant, demonstrating that angular momentum has been conserved by gravity. This satisfies the students as the traditional explanation for Kepler's Second Law follow angular momentum arguments.

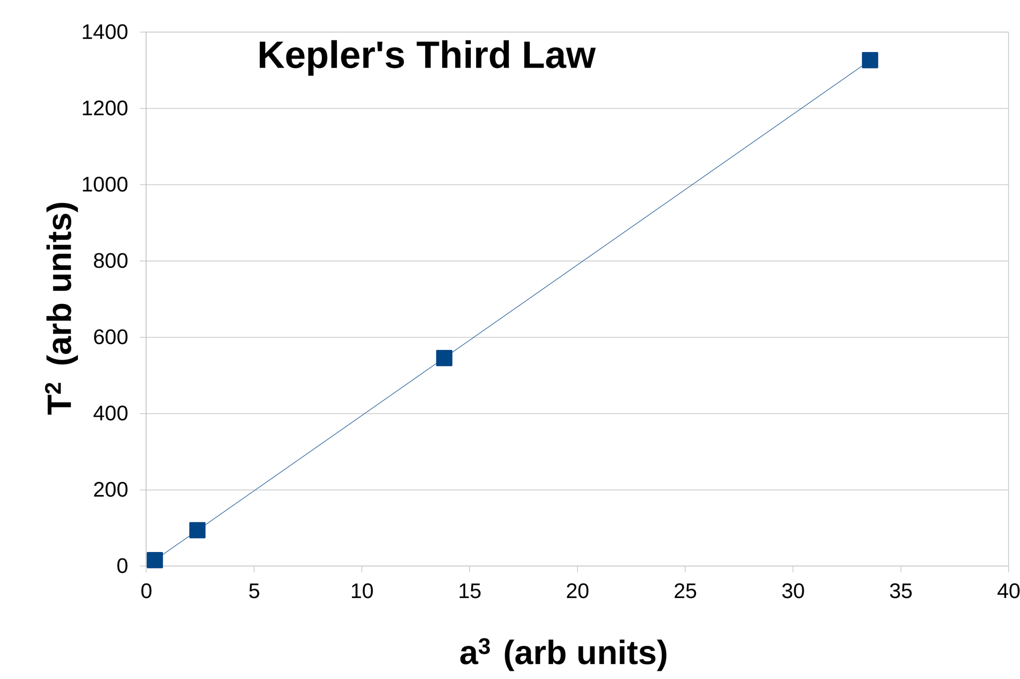
[1] R. P. Feynman, R.B. Leighton and M.L. Sands, *The Feynman Lectures on Physics*, Commemorative Issue (Addison-Wesley, Reading, 1989), p. 9-5 to 9-9.
[2] A. L. Garcia, *Numerical Methods for Physics*, 1st ed. (Prentice-Hall, Englewood Cliffs, 1994), p. 37.

Statements of Kepler's Laws from Wikipedia: Wikipedia contributors, "Kepler's laws of planetary motion," *Wikipedia, The Free Encyclopedia*, http://en.wikipedia.org/w/index.php?title=Kepler%27s_laws_of_planetary_motion&oldid=636349522 (accessed December 3, 2014).

The Third Law

III - *The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.*

The Third Law is usually demonstrated by assuming a circular orbit and equating the centripetal force with Newtonian gravity. This is dissatisfying as it gives the relationship $T^2 = r^3$, masking that the more complete treatment scales the square of the period with the cube of the semi-major axis. Once students have a working spreadsheet, they can vary the semi-major axis of their orbits by changing the initial conditions. By playing with the step size in time, they can determine how long it takes to complete one orbit. From this they can graph the two quantities:



This approach relaxes the circular constraint, as the data points in the graph can come from ellipses with a wide range of eccentricities, and captures the correct semi-major axis dependence.

More advanced students can be encouraged to investigate the relationship between the slope of the line and the value of the central mass. While it is useful in initial attempts at spreadsheets to set GM to unity, actual values can be used to model the orbits of planets around the sun or moons around a planet. When the true values are used, it is found that the slope is inversely proportional to the value of the central mass. Note that this model, with a fixed massive central object, does not give the correct M+m behavior for the constant of proportionality.